Problem Set 1

1. Find the value of the expression

$$2\sqrt{\frac{x-y}{x+y}}+\frac{2x\sqrt{x^2-y^2}}{y^2\left(xy^{-1}+1\right)^2}\cdot\frac{1}{1+\frac{1-yx^{-1}}{1+yx^{-1}}}$$

given that $x=\frac{26}{17}$ and $y=\frac{10}{17}$.
Answer: 2.
Solution. Simplifying the expression, we get

$$2\sqrt{\frac{x-y}{x+y}}+\frac{2x\sqrt{x^2-y^2}}{(x+y)^2}\cdot\frac{1+yx^{-1}}{2}=2\sqrt{\frac{x-y}{x+y}}+\frac{2x\sqrt{x^2-y^2}}{(x+y)^2}\cdot\frac{x+y}{2x}=$$
$$=2\sqrt{\frac{x-y}{x+y}}+\frac{\sqrt{x^2-y^2}}{x+y}=\backslash x+y>0\backslash=2\sqrt{\frac{x-y}{x+y}}+\sqrt{\frac{x-y}{x+y}}=3\sqrt{\frac{x-y}{x+y}}.$$

Substituting the values of x and y, we obtain $3\sqrt{\frac{16/17}{36/17}}=3\sqrt{\frac{4}{9}}=2.$

2. In a right triangle ABC with hypotenuse BC angle bisector BL and median AM are perpendicular to each other. Find $\sin(1111\angle ACB)$.
Answer: -0.5.
Solution. Let T be the intersection point of BL and AM. Line segment BT is both an angle bisector and an altitude in triangle ABM, therefore, triangle ABM is isosceles ($AB=BM$). According to the property of a median in a right triangle, $AM=BM=MC$. Hence, $AB=BM=AM$, i.e. triangle ABM is equilateral. Consequently, $\angle ABC=\frac{\pi}{3}$, $\angle ACB=\frac{\pi}{6}$. So $\sin(1111\angle ACB)=\sin\frac{1111\pi}{6}=-0.5$.

3. Solve the system of equations $\begin{cases}(x+y)(xy+1)=19,\\(x+3)(y+3)=-14.\end{cases}$ In the answer specify the largest possible value of the expression $6x+y$. If the system has no real solutions write 2024 instead.
Answer: 19.
Solution. The left-hand side of the second equation can be written as $xy+3(x+y)+9$. Introducing a substitution $x+y=u$, $xy=v$, we obtain

$$\begin{cases}u(v+1)=19,\\3u+v+23=0\end{cases}\Leftrightarrow\left[\begin{array}{l}u=-1,v=-20,\\u=-\frac{19}{3},v=-4.\end{array}\right.$$

If $u=-1$, $v=-20$, we get two solutions: $(-5;4)$ and $(4;-5)$. If $u=-\frac{19}{3}$, $v=-4$, we get two more solutions: $\left(\frac{-19-\sqrt{505}}{6};\frac{-19+\sqrt{505}}{6}\right)$ and $\left(\frac{-19+\sqrt{505}}{6};\frac{-19-\sqrt{505}}{6}\right)$.
The value of $6x+y$ for each of the solutions is equal to -26, 19, $\frac{-133-5\sqrt{505}}{6}$, $\frac{-133+5\sqrt{505}}{6}$ respectively. The largest of these is 19.

4. How many integer values of x such that $|x|<100$ satisfy the inequality

$$3\log_{27}(4x^2+1)\geq\log_3(3x^2+4x+1)?$$

Answer: 195.
Solution. The inequality is equivalent to the following one $\log_3\left(4x^2+1\right)\geq\log_3\left(3x^2+4x+1\right)$, which yields $4x^2+1\geq 3x^2+4x+1>0$. From here we find that $x\in(-\infty;-1)\cup(-1/3;0]\cup[4;+\infty)$. There are 195 integer values in the solution set which satisfy the inequality $|x|<100$.

5. Three cyclists are riding along a highway in the same direction, each traveling at a constant speed. At the moment when the first two cyclists were at the same point, the third was 6 km behind them. At the moment when the third cyclist caught up with the second, the first was 3 km behind them. How many kilometers was the second cyclist ahead of the first at the moment when the first and third were at the same point?
Answer: 2.
Solution. Let the cyclists' speeds be v_1, v_2, v_3. It follows from the task that $v_1 < v_2 < v_3$. The third one is riding $v_3 - v_2$ faster than the second one, so he needs a time of $\frac{6}{v_3-v_2}$ to catch up with the second one. During this time, the second one overtakes the first one by $\frac{6}{v_3-v_2} \cdot (v_2 - v_1)$, which is equal to 3 according to the task. From this we find that $v_3 = 3v_2 - 2v_1$.
The third one catches up with the first one after a time of $\frac{6}{v_3-v_1}$. During this time, the second one manages to move away from the first one by $\frac{6}{v_3-v_1} \cdot (v_2 - v_1) = \frac{6(v_2-v_1)}{3v_2-2v_1-v_1} = 2$. This means that the second cyclist is 2 km ahead when the third and first one meet.

6. How many integer values of parameter a are there such that the inequality $ax^2 + 4ax + 25 \leq 0$ has no real solutions?
Answer: 7.
Solution. There are no solutions if $a = 0$. If $a > 0$, there are no solutions when the discriminant is negative. Hence we obtain $\frac{D}{4} = 4a^2 - 25a < 0$, and so $0 < a < \frac{25}{4}$. If $a < 0$, then the solutions necessarily exist. Uniting all the results, we obtain $0 \leqslant a < \frac{25}{4}$. This interval has 7 integer values.

7. Solve the equation
$$\sin^2 2x + \sin^2 4x = 1 - \frac{\cos 2x}{\cos 3x}.$$
In the answer write down the number of its roots that satisfy the inequality $-2\pi \leqslant x \leqslant \frac{9\pi}{2}$.
Answer: 23.
Solution. The initial equation is equivalent to each of the following ones:
$$\frac{1-\cos 4x}{2} + \frac{1-\cos 8x}{2} = 1 - \frac{\cos 2x}{\cos 3x},$$
$$\frac{\cos 4x + \cos 8x}{2} = \frac{\cos 2x}{\cos 3x}, \quad \cos 6x \cos 2x = \frac{\cos 2x}{\cos 3x},$$
and under condition $\cos 3x \neq 0$ it is equivalent to the equation
$$\cos 2x(\cos 3x \cdot \cos 6x - 1) = 0.$$
From here we have
$$\left[\begin{array}{l}\cos 2x = 0, \\ \frac{\cos 3x+\cos 9x}{2} = 1\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\cos 2x = 0, \\ \cos 3x = \cos 9x = 1\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\cos 2x = 0, \\ \cos 3x = 1\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x = \frac{\pi}{4} + \frac{\pi k}{2}, \\ x = \frac{2\pi k}{3}, k \in \mathbb{Z}.\end{array}\right.$$
It is easy to check thatfor all the values of x we obtainedthe inequality $\cos 3x \neq 0$ holds.
For finding the number of roots we start with making sure that two sets of roots do not have intersections:
$$\frac{\pi}{4} + \frac{\pi k}{2} = \frac{2\pi n}{3} \Leftrightarrow \frac{3}{2} + 3k = 4n,$$
which is impossible, since the left-hand side is a fraction, and the right-hand side is an integer. Therefore,
$$-2\pi \leqslant \frac{\pi}{4} + \frac{\pi k}{2} \leqslant \frac{9\pi}{2} \Leftrightarrow -\frac{9}{2} \leqslant k \leqslant \frac{17}{2} \Leftrightarrow -4 \leqslant k \leqslant 8 \Rightarrow 13 \text{ roots};$$
$$-2\pi \leqslant \frac{2\pi k}{3} \leqslant \frac{9\pi}{2} \Leftrightarrow -3 \leqslant k \leqslant \frac{27}{4} \Leftrightarrow -3 \leqslant k \leqslant 6 \Rightarrow 10 \text{ roots}.$$
All in all, there are 23 roots on the interval.

8. Let G be the midpoint of hypotenuse PR of the right triangle PQR. Line ℓ passes through point G and intersects with side QR at point A, and with the extension of side PQ beyond point P at point B. Find the area of triangle PQR if it is known that $AR = 10$, $BP = 20$, $\angle RPQ = \arccos\frac{3}{5}$.
Answer: 384.
Solution. Let us choose point C on side PR in such a way that $AC \parallel PQ$; then we have that $CR = \frac{25}{2}$, $AC = \frac{15}{2}$. Let $GP = GR = x$; from the similarity of triangles ACG and BPG it follows that $CG : PG = AC : PB$, i.e. $(x - \frac{5}{4}) : x = \frac{3}{4} : 2$, from where we get that $x = 20$. Consequently, $PR = 40$, $A = \frac{1}{2}\,PQ \cdot QR = \frac{1}{2}\,PR^2\,\sin\alpha \cdot \cos\alpha = 384$, for $\sin\alpha = \frac{AC}{CR} = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$.

Problem Set 2.

1. Find the value of the expression

$$\left[1-\left(\frac{a^{-0.75}+1}{a^{-0.25}+1}+\frac{3}{a^{0.25}}\right):\left(a^{-0.25}+1\right)\right]:a^{-0.75}$$

given that $a=289$.

Answer: -17.

Solution. If we denote $a^{-0.25}=t$, the expression becomes

$$\left[1-\left(\frac{t^3+1}{t+1}+3t\right):(t+1)\right]:t^3=\left[1-\left(t^2+2t+1\right):(t+1)\right]:t^3=[1-(t+1)]:t^3=-\frac{1}{t^2}.$$

Going back to variable a, we obtain $-\sqrt{a}$. As $a=289$, the result is -17.

2. Line segment BH is the altitude in an acute triangle ABC. It is known that $AB=2CH$ and $CB=2AH$. Find $\cos(2024\angle ABC)$.

Answer: 0.5.

Solution. Let $AB=2CH=2x$ and $CB=2AH=2y$. Applying Pythagorean theorem, we obtain $BH^2=AB^2-AH^2=BC^2-CH^2=(2x)^2-y^2=(2y)^2-x^2$. Thus $x=y$, and ABC is an equilateral triangle. Therefore, $\cos(2024\angle ABC)=\cos\frac{2024\pi}{3}=-0.5$.

3. Two marksmen fired 60 shots each, with a total of 99 misses and 21 hits. It is known that the first marksman had t hits for every miss, and the second marksman had $3t$ hits for every miss (the value of t is not specified). How many hits did the second marksman make?

Answer: 15.

Solution. Let the first marksman have x misses. Then the second marksman has $(99-x)$ misses. The number of hits is tx for the first marksman and $3t(99-x)$ for the second one. Since each of them made 60 shots, we obtain a system of equations $x+tx=60$, $99-x+3t(99-x)=60$, solving which we find that $t=\frac{1}{9}$, $x=54$. Hence the number of hits for the second marksman is $3t(99-x)=15$.

4. The sum of digits in a three-digit number is equal to 15. The sum of squares of the digits in the number is equal to 93. If a number written with the same digits in the opposite order is subtracted from the initial one, the result is 297. What is the smallest possible value of the initial number?

Answer: 582.

Solution. Let $\overline{abc}$ be the given three-digit number. Then it follows from the task that

$$\begin{cases} a+b+c=15, \\ (100a+10b+c)-(100c+10b+a)=297, \\ a^2+b^2+c^2=93. \end{cases}$$

This system of equations has two solutions: $a=8$, $b=2$, $c=5$ and $a=5$, $b=8$, $c=2$. The corresponding numbers are 825 and 582. The smallest of the two is 582.

5. Solve the equation $\sqrt{24-12x}+2\sqrt{11-8x}=5$. If there is one root write it down in the answer. If there are several roots write their sum. If there are none write 2024.

Answer: 1.25.

Solution. The domain of the equation is $x\leq\frac{11}{8}$. On the domain it can be transformed in the following way:

$$2\sqrt{11-8x}=5-\sqrt{24-12x}\Rightarrow 44-32x=25+24-12x-10\sqrt{24-12x}\Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{24-12x}=1+4x\Rightarrow 96-48x=16x^2+8x+1\Leftrightarrow\left[\begin{array}{l} x=1.25, \\ x=-4.75. \end{array}\right.$$

Both values of x belong to the domain, but when substituting them into the initial equation, we can see that only $x=1.25$ satisfies it.

6. Find the sum S of all the roots of the equation

$$\frac{\cos^3 x\ \sin 3x}{\sin x} + \sin^2 x\ \cos 3x = 6\ \cos 2x\ \cos^2 x$$

that satisfy the inequality $-\frac{\pi}{2} \leq x \leq \frac{5\pi}{2}$. In the answer write down the value of $\frac{S}{\pi}$.

Answer: 14.

Solution. The left-hand side of the equation can be transformed as follows:

$$\frac{\cos^3 x\ \sin 3x}{\sin x} + \sin^2 x\ \cos 3x = \frac{\cos^3 x\left(3\sin x - 4\sin^3 x\right)}{\sin x} + \sin^2 x\left(4\cos^3 x - 3\cos x\right) =$$
$$= \cos^3 x\left(3 - 4\sin^2 x\right) + \left(1-\cos^2 x\right)\left(4\cos^3 x - 3\cos x\right) =$$
$$= \cos^3 x\left(4\cos^2 x - 1\right) + \left(1-\cos^2 x\right)\left(4\cos^3 x - 3\cos x\right) =$$
$$= 6\cos^3 x - 3\cos x = 3\cos x\left(2\cos^2 x - 1\right) = 3\cos x\cos 2x.$$

Thus under condition that $\sin x \neq 0$ the equation is equivalent to the following:

$$3\cos x\cos 2x = 6\cos 2x\cos^2 x \Leftrightarrow \left[\begin{array}{l}\cos 2x = 0,\\ \cos x = 0,\\ \cos x = \frac{1}{2}\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x = \frac{\pi}{4} + \frac{\pi k}{2},\\ x = \frac{\pi}{2} + \pi k,\\ x = \pm\frac{\pi}{3} + 2\pi k, k \in \mathbb{Z}.\end{array}\right.$$

Obviously, all the roots obtained satisfy the restriction $\sin x \neq 0$.
The value of S is equal to

$$-\frac{\pi}{2} - \frac{\pi}{3} - \frac{\pi}{4} + \frac{\pi}{4} + \frac{\pi}{3} + \frac{\pi}{2} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{3\pi}{2} + \frac{5\pi}{3} + \frac{7\pi}{4} + \frac{9\pi}{4} + \frac{7\pi}{3} + \frac{5\pi}{2} = 14\pi.$$

So $\frac{S}{\pi} = 14$.

7. How many integer values of parameter a are there such that the equation $(a^2-9)x^2+5 = 2\sqrt{a+3}\cdot x$ has at least one real solution?

Answer: 6.

Solution. Moving all the terms to the left-hand side, we get an equation $(a^2-9)x^2-2\sqrt{a+3}\cdot x+5 = 0$. For the square root to exist, the inequality $a \geq -3$ has to hold.

- If $a = 3$ we get an equation $-2\sqrt{6}x + 5 = 0$, which has a solution.
- If $a = -3$ there are no solutions.
- And if $a \neq \pm 3$, the equation is quadratic. For it to have at least one real root, its discriminant has to be non-negative. We obtain $\frac{D}{4} \geq 0 \Leftrightarrow a+3-5(a^2-9) \geq 0$. Therefore, $a \in \left[-3; \frac{16}{5}\right], a \neq \pm 3$.

Uniting the solutions obtained, we get that the set of values of parameter that satisfy the task is $a \in \left(-3; \frac{16}{5}\right]$. There are six integer values in it.

8. Circle ω is inscribed into an angle with vertex C. The circle is tangent to the sides of the angle at points A and B. Point K is chosen on the arc of circle ω situated outside of the triangle ABC. The distances from K to lines AC and BC are equal to 39 and 156 respectively. Find the distance from point K to line AB.

Answer: 78.

Solution. Let E, F and M be the bases of the perpendiculars dropped from point K onto lines BC, AC and AB respectively. As $\angle KBE = \angle KAB$, triangles KAM and KBE are similar to each other, consequently,

$$\frac{KM}{KA} = \frac{KE}{KB}, \tag{1}$$

where $KE = 156$. In the same way, from the similarity of triangles KAF and KBM it follows that

$$\frac{KM}{KB} = \frac{KF}{KA}, \tag{2}$$

where $KF = 39$. Multiplying the equations (1) and (2) yields

$$KM^2 = KE \cdot KF = 39 \cdot 156,$$

hence, $KM = 78$.

Problem Set 3

1. Find the sum of all the roots of the equation $\dfrac{3}{3+\sqrt{2x}}-\dfrac{1}{4}=\dfrac{5}{\sqrt{18x}+2x}$. If there are no roots write 2024 in the answer.
   Answer: 20.5.
   Solution. Multiplying both sides of the equation by $4\sqrt{2x}\left(3+\sqrt{2x}\right)$, we get $12\sqrt{2x}-20=3\sqrt{2x}+2x$, $2x-9\sqrt{2x}+20=0$. Solving the equation as quadratic with respect to $\sqrt{2x}$, we find $\sqrt{2x}=5$ or $\sqrt{2x}=4$, and so $x=12.5$ or $x=8$. Both roots belong to the domain, and their sum is equal to 20.5.

2. How many integer values of x are there that satisfy the inequality

$$1+\log_7(x+10)\le\log_7(100-x^2)?$$

   If there are infinitely many of them write 20.24 in the answer.
   Answer: 13.
   Solution. The inequality is equivalent to the following one: $\log_7(7(x+10))\le\log_7(100-x^2)$, which yields $0<7(x+10)\le 100-x^2$. Its solution set is $x\in(-10;3]$. It contains 13 integer values.

3. Perpendicular NP is dropped from point N on side BC of the equilateral triangle ABC onto its side AB. The circumcircle ω of triangle BNP is tangent to line AN. Find the radius of ω if the perimeter of triangle ABC is equal to 15.
   Answer: 1.25.
   Solution. Triangle BNP is right, so its hypotenuse BN is a diameter of its circumcircle ω. As ω is tangent to line AN, diameter BN is perpendicular to this line. Hence, AN is an altitude of an equilateral triangle ABC. Consequently, N is a midpoint of BC. From here it follows that $BN=\frac{BC}{2}=\frac{15}{6}=2.5$. The radius in question is equal to 1.25.

4. The tornado is moving in a straight line. At noon, the center of the tornado was 24 km to the east and 5 km to the south of the village administration, and twenty minutes later — 20 km to the east and $\frac{10}{3}$ km to the south of the village administration. At what minimum distance from the village administration will the center of the tornado pass? Express your answer in meters and round it to the nearest integer.
   Answer: 4 615.
   Solution. Let us introduce a coordinate system with the origin at the village administration. Let the x-axis be directed to the east and the y-axis to the north. Then the two tornado location measurements correspond to the points with coordinates $(24;-5)$ and $\left(20;-\frac{10}{3}\right)$. The equation of the line passing through these points is $5x+12y-60=0$. The distance from the origin to this line is $\frac{|5\cdot0+12\cdot0-60|}{\sqrt{5^2+12^2}}=\frac{60}{13}$ km. Expressing this distance in meters, we have $\frac{60}{13}\cdot1000\approx4\,615$ meters.

5. There are two sequences: an arithmetic sequence $\{a_n\}$ and a geometric sequence $\{b_n\}$. It is known that $a_1=b_1$, $a_2=b_2$, $a_4=b_3$. What is the largest possible value of the common ratio of $\{b_n\}$ given that the common difference of $\{a_n\}$ is distinct from zero?
   Answer: 2.
   Solution. If d is the common difference of $\{a_n\}$ and q is the common ratio of $\{b_n\}$, we get from the task that $a_1=b_1$, $a_1+d=b_1q$, $a_1+3d=b_1q^2$. Dividing the second equation by the first, and the third by the second, we get

$$\frac{a_1+d}{a_1}=q,\qquad\frac{a_1+3d}{a_1+d}=q.$$

   Equating the left sides of the equations and simplifying, we obtain $d^2=a_1d$, and so $a_1=d$ for $d\ne0$. Consequently, $q=\frac{d+d}{d}=2$.

6. Solve the equation

$$\left(\sqrt{3}\cos 2x+\sin 2x\right)^2=7+3\cos\left(2x-\frac{\pi}{6}\right).$$

In the answer write down the sum of roots that belong to the interval $12\pi \leqslant x \leqslant 30\pi$, divided by π.
Answer: 379.5.
Solution. Let $\cos\left(2x-\frac{\pi}{6}\right)=t$, then $\sqrt{3}\cos 2x+\sin 2x=2\cos\left(2x-\frac{\pi}{6}\right)=2t$ the equation takes the form of $4t^2-3t-7=0$, and from here $t_1=-1$, $t_2=\frac{7}{4}$. Consequently, $\cos\left(2x-\frac{\pi}{6}\right)=-1$, and $2x-\frac{\pi}{6}=\pi+2\pi n$, $x=\frac{7\pi}{12}+\pi n$, $n\in\mathbb{Z}$. For the roots to belong to the interval $[12\pi;30\pi]$, the inequality $12\pi \leqslant \frac{7\pi}{12}+\pi n \leqslant 30\pi$ has to hold, and from here $12 \leqslant n \leqslant 29$. Hence the sum of the roots is

$$\left(\frac{7\pi}{12}+12\pi\right)+\left(\frac{7\pi}{12}+13\pi\right)+\ldots+\left(\frac{7\pi}{12}+29\pi\right)=379.5\pi.$$

7. What is the largest value of parameter t such that the system of equations

$$\begin{cases} |x-2|+2|y|=4, \\ x^2+y^2=2x+t \end{cases}$$

has exactly three real solutions $(x;y)$?
Answer: 8.
Solution. Let us rewrite the second equation as $(x-1)^2+y^2=t+1$. The graph of the first equation is a rhombus with vertices $(-2;0)$, $(6;0)$, $(2;2)$, $(2;-2)$. The graph of the second equation for $t>-1$ is a circle centered at $(1;0)$ with radius $\sqrt{t+1}$ ($t=-1$ yields a degenerate case of a point $(1;0)$, and if $t<-1$ the equation corresponds to an empty set). Both sets are symmetric with respect to x-axis; hence the system can have three solutions (which is an odd number of solutions) only if one of the solutions belongs to x-axis. It happens when the circle intersects with the rhombus either at $(-2;0)$ or at $(6;0)$. In the first case $t=8$ and the system has three solutions, and in the second case $t=24$, and the system has exactly one solution.

8. In a trapezoid $ABCD$ with bases AD and BC angle ADC is right. Point S is chosen on a line segment BD in such a way that $BS:SD=1:3$. Circle ω centered at S intersects with line BC at points P and M and is tangent to the straight line AD. Find the length of AB given that $BC=9$, $AD=8$, $PM=4$.
Answer: 3.
Solution. Let us denote $CD=x$, and let B' and S' be projections of points B and S onto the straight line AD, and S'' the projection of S onto the straight line BC. Then we have that $SS'=SM=\frac{3}{4}x$, $SS''=\frac{1}{4}x$, $S''M=\sqrt{(SM)^2-(SS'')^2}=\frac{x}{\sqrt{2}}$, $PM=2S''M=x\sqrt{2}=4$, and so $x=2\sqrt{2}$. Hence

$$AB=\sqrt{(BB')^2+(B'A)^2}=\sqrt{(CD)^2+(BC-AD)^2}=3.$$

Problem Set 4.

1. Let CD be the altitude of a right triangle ABC with hypotenuse AB and $\angle BAC = \frac{\pi}{6}$. Find the sum of legs of triangle ABC given that $BD + CD = 2\,024$.
   Answer: 4 048.
   Solution. Triangles ABC and CBD are similar, and the coefficient of similarity is 2. Hence the sum in question is equal to $2\,024 \cdot 2 = 4\,048$.

2. How many integer values of parameter a are there such that the inequality $x^2 + (a-1)x + 4 \geq 0$ holds for all real values of x? If there are infinitely many of them write 20.24 in the answer.
   Answer: 9.
   Solution. The inequality holds for all real values of x if and only if its discriminant is non-positive, i.e. $D \leq 0$. Thus $D = (a-1)^2 - 16 \leq 0$, and so $|a-1| \leq 4$. This inequality has 9 integer solutions.

3. How many integer values of x are there such that $|x| < 10$ and they satisfy the inequality
$$\log_2(x-2)^2 + 2\log_2(x+4) \geq 6?$$
   Answer: 9.
   Solution. The inequality is equivalent to the following one: $\log_2(|x-2|(x+4)) \geq 3$. If $-4 < x < 2$ then $-(x-2)(x+4) \geq 8$, and from here $-2 \leq x \leq 0$. If $x > 2$ then $(x-2)(x+4) \geq 8$, and so $x \geq \sqrt{17} - 1$. There are 9 integer solutions of the inequality that satisfy $|x| < 10$.

4. Three cyclists are riding along a highway from point A to point B. Two of them leave at the same time, with the first riding at 15 km/h and the second at 12 km/h. The third cyclist leaves after 45 minutes, catching up with the first 52.5 minutes later than the second. What is the speed of the third cyclist? Express this speed in km/h.
   Answer: 9.
   Solution. Let the speed of the third cyclist be x km/h. In 45 minutes the second cyclist manages to travel $\frac{3}{4} \cdot 12 = 9$ km. The third cyclist travels at a speed $(x-12)$ km/h greater than the second. This means that in order to catch up with the second by 9 km, he will need $\frac{9}{x-12}$ h.
   Arguing similarly, we get that he needs $\frac{45}{4(x-15)}$ h to catch up with the first. By the task, the first time is less than the second by 52.5 minutes, i.e. by $\frac{7}{8}$ hours, from which we obtain the equation
$$\frac{45}{4(x-15)} = \frac{9}{x-12} + \frac{7}{8}.$$
   Its solutions are $x = 21$ and $x = \frac{60}{7}$. The second root does not satisfy the conditions of the problem, since in order for the third cyclist to catch up with the first two, his speed must be greater than 15 km/h.

5. Find all pairs of integers $(x;y)$ that satisfy both inequalities $2x^2 + 2y^2 + 12y + 65 < 20x$ and $x + 3y + 3 < 0$. In the answer write down the number of pairs.
   Answer: 3.
   Solution. Completing the squares in the left-hand side of the first inequality, we obtain $2(x-5)^2 + 2(y+3)^2 < 3$. As x and y are integers, there are only three options:

   - $(x-5)^2 = (y+3)^2 = 0 \quad \Longrightarrow \quad x = 5,\ y = -3,$
   - $(x-5)^2 = 1,\ (y+3)^2 = 0 \quad \Longrightarrow \quad \left[\begin{array}{l} x = 6,\ y = -3, \\ x = 4,\ y = -3, \end{array}\right.$
   - $(x-5)^2 = 0,\ (y+3)^2 = 1 \quad \Longrightarrow \quad \left[\begin{array}{l} x = 5,\ y = -2, \\ x = 5,\ y = -4. \end{array}\right.$

   Now we need to check which of these satisfy the second inequality. They are $(5;-3)$, $(4;-3)$, $(5;-4)$.

6. Solve the equation

$$\frac{\operatorname{ctg} x \ \operatorname{ctg} 2x}{\operatorname{ctg} 2x - \operatorname{ctg} x} = \frac{\operatorname{ctg} 4x \ \operatorname{ctg} 3x}{\operatorname{ctg} 4x - \operatorname{ctg} 3x}.$$

In the answer write down the sum of the roots that satisfy the inequality $-3\pi \leqslant x \leqslant 10\pi$.

Answer: 52.

Solution. Let us transform the denominators of both sides of the equation:

$$\operatorname{ctg} 2x - \operatorname{ctg} x = \frac{\cos 2x}{\sin 2x} - \frac{\cos x}{\sin x} = -\frac{\sin x}{\sin x \ \sin 2x} = -\frac{1}{\sin 2x}, \quad \sin x \neq 0;$$

$$\operatorname{ctg} 4x - \operatorname{ctg} 3x = \frac{\cos 4x}{\sin 4x} - \frac{\cos 3x}{\sin 3x} = -\frac{\sin x}{\sin 3x \ \sin 4x}.$$

The domain of the equation is defined by the following restrictions

$$\sin x \ \sin 2x \ \sin 3x \ \sin 4x \neq 0. \qquad (1)$$

On its domain the initial equation is equivalent to the following ones:

$$\frac{\cos x \ \cos 2x \ \sin 2x}{\sin x \ \sin 2x} = \frac{\cos 3x \ \cos 4x \ \sin 3x \ \sin 4x}{\sin 3x \ \sin 4x \ \sin x},$$

$$\cos x \ \cos 2x = \cos 3x \ \cos 4x, \quad \cos 3x + \cos x = \cos 7x + \cos x,$$

$$\cos 3x - \cos 7x = 0, \quad \sin 5x \ \sin 2x = 0.$$

As $\sin 2x \neq 0$, we get $\sin 5x = 0$, откуда $x = \pi n/5$, $n \in \mathbb{Z}$. The condition (1) holds if $n \neq 5k$, $k \in \mathbb{Z}$. It remains to find the number of roots that satisfy the inequality.

$$-3\pi \leqslant \frac{\pi n}{5} \leqslant 10\pi \quad \Leftrightarrow \quad -15 \leqslant n \leqslant 50 \quad \Rightarrow \quad 66 \text{ values } n,$$

$$\text{out of them 14 are divisible by 5} \quad \Rightarrow \quad 66 - 14 = 52 \text{ roots on the interval.}$$

7. During a month, the store sold 21 tons of candies of three types. It is known that candies of the first type cost 600 rubles per kilogram, of the second type 400 rubles, and of the third type 200 rubles. The masses of candies sold make up a geometric sequence in the order indicated above, and the total amount received for the candies was 6, 600, 000 rubles. How many kilograms of the candies of the first type were sold?

Answer: 3 000.

Solution. Let's say that x kilograms of candies of the first type were sold, xq of the second type, and xq^2 of the third type. From the task we obtain that $x+xq+xq^2 = 21\,000$ and $600x+400xq+200xq^2 = 6\,600\,000$. Dividing the second equation by the first, we have $\frac{3+2q+q^2}{1+q+q^2} = \frac{11}{7}$; solving the latter equation yields $q = 2$ (the second root of the equation is negative). Then $x = \frac{21\,000}{1+q+q^2} = 3\,000$. This is the answer to the problem.

8. Let ω be the circumcircle of triangle PQR, and G and H respectively be the intersection points of the extensions of medians QA and PB with the circumcircle of triangle PQR. Find the radius of ω provided that $QA = AG$, $PH : PB = 3 : 2$, and the area of triangle PQR is equal to 80.

Answer: 10.

Solution. As chords PR and QG bisect each other at their intersection point A, quadrilateral $QRGP$ is a parallelogram inscribed into a circle. Thus it is a rectangle, and so $\angle PQR = \frac{\pi}{2}$ and A is the center of the circle ω. Let $BH = x$, then $BP = 2x$. According to the intersecting chords theorem $BP \cdot BH = BQ \cdot BR$. But $BR = BQ$, and thus $BQ^2 = 2x^2$, $BQ = x\sqrt{2}$. From a right triangle BPQ we get that $PQ = \sqrt{BP^2 - BQ^2} = x\sqrt{2}$. All in all, the legs of a triangle PQR are equal to $x\sqrt{2}$ and $2x\sqrt{2}$, and its hypotenuse is equal to $x\sqrt{10}$. The area of triangle PQR is equal to $\left(x\sqrt{2}\right)^2 = 80$, $x = 2\sqrt{10}$. The diameter of ω is equal to the length of the hypotenuse, i.e. to 20. Therefore, the radius of ω is 10.